

Lower bound to the mass of a relativistic three-boson system in the light-cone

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Abstract. The lower bound masses of the ground-state relativistic three-boson system in 1+1, 2+1 and 3+1 space-time dimensions are obtained. We have considered a reduction of the ladder Bethe-Salpeter equation to the light-front in a model with renormalized two-body contact interaction. The lower bounds are deduced with the constraint of reality of the two-boson subsystem mass. It is verified that, in some cases, the lower bound approaches the ground state binding energy. The corresponding non-relativistic limits are also verified.

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1. Introduction

Recent applications of concepts of effective field theory to Quantum Chromodynamics, in which the effective degrees of freedom are the constituent quarks, and the baryons are viewed as bound system of three relativistic constituent quarks [1, 2], make it worthwhile to study the properties of relativistic equations of three-particles interacting through effective attractive forces. The schematic model, represented by a contact interaction, gives one of the phenomenological possibilities; even lacking confinement it can represent the relativistic propagation of the system in a domain inside the confinement region. The relativistic equations can be reduced to the light-front [3, 4], allowing a definition of a light-front wave-function on the null-plane hypersurface ($x^+ = t + z = 0$). In the three-body case, the lower Fock-state component is given by a three-particle wave-function. The radial wave-function of the baryon, for light-quarks, is simplified by a

totally symmetric component. Disregarding spin effects, it can be represented by a three-boson wave-function. To study the general properties of the three-boson Bethe-Salpeter equation, we allow the space-time dimensions to run from two to four ($n=2, 3$ and 4), as in each case there are different physical aspects related to the dimensions that are relevant to address.

The relativistic light-front representation of the three-body bound-state wave-function are well known to contain at least two convenient and simple characteristics [2, 5, 6, 7]:

- a) the center-of-mass coordinate (CM) can be easily separated;
- b) the wave-function is covariant under kinematical boosts.

These features allow one to calculate some observables with the bound state wave-function described only in terms of particle degrees of freedom. Light-front wave-functions have been extensively used to construct relativistic models for the light pseudoscalar mesons [8, 9, 10, 11, 12], vector mesons [13, 14, 15] and for the nucleon [1, 2, 16, 17, 18, 19], in terms of constituents quark bound systems.

The particular aspect that we are concerned in this work is the ground-state mass of the three-boson system in 2, 3 and 4 space-time dimensions. The approach that we have considered consists of a relativistic Bethe-Salpeter equation reduced to the light-front, with renormalized two-body contact interaction, parametrized by the two-boson binding energy. This model has been applied with some success to describe the charge distribution of the nucleon in terms of the constituent quarks [19]. Here, we derive a lower bound to the ground state binding energy of a three-boson system, independent of the dimension, generalizing the findings of reference [3], and compare with the numerical results from the solution of the Bethe-Salpeter equation. The approach does not have retardation effects, which simplifies the relativistic dynamical description of the bound state system. The non-relativistic limit of the equation is also studied, since it explains the results for the two-boson binding energies close to zero.

The paper is organized as follows: In section 2, we present the covariant ladder Bethe-Salpeter equation for the Faddeev component of the three-boson vertex in n -dimensions (Faddeev Bethe-Salpeter equation), using a zero-range two-body interaction (a constant in momentum space). In a subsection, we also obtain explicitly the corresponding two-body scattering amplitude, in the ladder approximation, for space-time dimensions from two to four. As we have indicated in this section and, subsequently, in section 3, the generalization of the approach from zero-range to a finite range interaction is not difficult. In section 3, we perform the reduction of the equations to the light-front, and obtain the corresponding non-relativistic limits. To make the formalism more general, allowing the inclusion of finite range effects, we include a single term separable interaction in the light-front form of the equations for the vertex function. We also discuss, in this section, the constraint in the phase-space of the spectator particle

momentum, due to the reality of the two-boson mass subsystem. In section 4 we obtain the lower bounds to the masses of the three-boson system (in the three cases we are considering). They are shown to be related to the restricted phase-space of the spectator particle. Finally, in section 5, we present our numerical results and concluding remarks.

2. Faddeev Bethe-Salpeter equation

The general features of the relativistic three-body equation projected in the light-front have been discussed in ref. [6], where it was implicitly suggested a calculation with a specific interaction. Therefore, it is instructive and illustrative to work out a simple dynamical model, like the one we present here, which can be easily compared with the corresponding covariant and non-relativistic approaches. The model can also be considered in its present form to define the dynamics in relativistic systems in two, three and four space-time dimensions.

The Faddeev-like equation for the component of the three-boson vertex is represented diagrammatically in figure 1, where the factor two in the rhs is related to the symmetrization of the total vertex. The vertex v , in the center-of-mass system, depends only on the energy-momentum vector of the spectator particle, $q \equiv q^\mu$. The three-boson vertex is the sum of the three components in which one of the bosons is the spectator. The two-boson scattering amplitude, $\tau(M_2)$, enters in the kernel of the integral equation for the vertex. From figure 1, we obtain

$$v(q^\mu) = 2\tau(M_2) \int \frac{d^n k}{(2\pi)^n} \left[\frac{i}{k^2 - M^2 + i\varepsilon} \right] \left[\frac{i}{(P_3 - q - k)^2 - M^2 + i\varepsilon} \right] v(k^\mu), \quad (1)$$

where n is the dimension of the space-time, P_3^μ is the three-boson energy-momentum vector in the center-of-mass system ($P_3^0 = M_{3B}$ and $P_3^{\mu \neq 0} = 0$) and the mass of the two-boson subsystem is given by $M_2^2 = (P_3 - q)^2$. In the present approach, the space-time dimension n can be two, three and four, such that the energy-momentum vector $q \equiv q^\mu$ is defined as (q^0, q^3) for $n = 2$, (q^0, q^3, q_\perp) for $n = 3$, and $(q^0, q^3, \vec{q}_\perp)$ for $n = 4$. \vec{q}_\perp is a $(n - 2)$ dimensional vector, that is zero for $n = 2$.

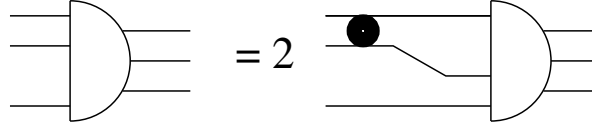


Figure 1. Diagrammatic representation of the integral equation for the Faddeev component of the vertex of three-boson bound-state.

In the following subsection, the two-boson scattering amplitude $\tau(M_2)$ that appears

in eq. (1) is explicitly obtained in the light-front for two, three and four dimensions.

2.1. Two-boson scattering amplitude

The diagram in figure 2 shows the amplitude for the two-boson scattering that, for a separable model interaction (as the present zero-range model) can be solved analytically. The solution in the center-of-mass system is given by

$$\tau^{-1}(M_2) = i \left\{ \lambda^{-1} + iB(M_2) \right\}, \quad (2)$$

where λ is the coupling constant of the interaction and M_2 the mass of the two boson system. The factor i , in eq. (2), comes from the S -matrix quantum field description of the two-body scattering process, which is represented in Fig. 2 by the corresponding Feynman diagram. $B(M_2)$ is the kernel of the scattering amplitude integral equation, represented by the bubble diagram of figure 3, that is given by

$$B(M_2) = - \int \frac{d^n k}{(2\pi)^n} \frac{1}{\{(k^2 - M^2 + i\varepsilon)((P - k)^2 - M^2 + i\varepsilon)\}}, \quad (3)$$

where M is the boson mass, $P^0 = M_2$ and $P^{\mu \neq 0} = 0$.

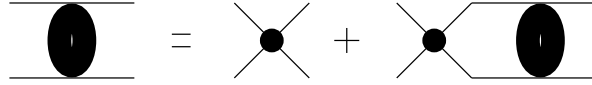


Figure 2. Diagrammatic representation of the integral equation for the two boson scattering amplitude.

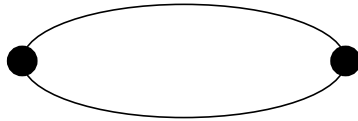


Figure 3. Bubble diagram corresponding to the kernel of the integral equation (2).

The momentum loop integral in eq. (3) is integrated in the null-plane energy $k^- (= k^0 - k^3)$, $k^+ (= k^0 + k^3)$ and \vec{k}_\perp :

$$B(M_2) = - \frac{1}{2(2\pi)^n} \int \frac{dk^- dk^+ d^{(n-2)}k_\perp}{k^+(P^+ - k^+)} \left(k^- - \frac{k_\perp^2(1 - \delta_{n,2}) + M^2 - i\varepsilon}{k^+} \right)^{-1}$$

$$\times \left(P^- - k^- - \frac{(P - k)_\perp^2 (1 - \delta_{n,2}) + M^2 - i\varepsilon}{P^+ - k^+} \right)^{-1}, \quad (4)$$

where the Kronecker symbol ($\delta_{i,j} = 1, \delta_{i,j \neq i} = 0$) was introduced to take into account the two-dimensional case.

The finite integration over k^- is performed with the use of Cauchy theorem. The position of the poles in the k^- complex-plane depends on the value of k^+ . For $k^+ > P^+$ and $k^+ < 0$, the poles are located in the same semi-plane, thus they don't contribute to the integration. Only if $0 < k^+ < P^+$ the integration is non-vanishing. We use the fraction of momentum $x = k^+/P^+$, within the interval $0 < x < 1$. This condition yields the suppression of the antiparticle propagation in the loop integral.

The result for $B(M_2)$ is given by

$$iB(M_2) = \frac{1}{2(2\pi)^{(n-1)}} \int \frac{dx}{x(1-x)} \int \frac{d^{(n-2)}k_\perp}{M_2^2 - \frac{k_\perp^2(1 - \delta_{n,2}) + M^2}{x(1-x)}}. \quad (5)$$

In the above equation, one may introduce a form factor to regularize the transverse integration in four space-time dimensions. In order to keep the Lorentz scalar property of the function $B(M_2)$, the form-factor can be a function of the free mass of the two-boson intermediate state that, in terms of the center-of-mass momentum, is written as

$$M_{02}^2 = 4(\vec{k}^2 + M^2) = \frac{k_\perp^2(1 - \delta_{n,2}) + M^2}{x(1-x)}, \quad (6)$$

and

$$iB(M_2) = \frac{1}{2(2\pi)^{(n-1)}} \int \frac{dx}{x(1-x)} \int d^{(n-2)}k_\perp \frac{g^2(M_{02})}{M_2^2 - M_{02}^2}. \quad (7)$$

A separable finite-range interaction between the two bosons is characterized by the form-factor $g(M_{02})$, which is equal to one in the zero-range limit.

The value of λ is chosen such that the two-boson system has one bound-state. So, a pole occurs in the scattering amplitude at the bound-state mass, where

$$\lambda^{-1} = -iB(M_{2B}); \quad (8)$$

and, from eq. (2), we obtain

$$\tau^{-1}(M_2) = B(M_{2B}) - B(M_2). \quad (9)$$

This result allows one to use $g \equiv 1$ in all the cases we are considering ($n = 2, 3$ and 4), because the function $B(M_2)$ is finite for $n = 2$ and $n = 3$; and, for $n = 4$, the divergence observed in eq. (5) is canceled. In four-dimensions, with $g = 1$, the integral has a logarithmic divergence that can be absorbed in the redefinition of λ , as in eq.(8).

The physical information introduced in the renormalization of the physical amplitude is the mass of the two bound bosons, M_{2B} .

The integration of $B(M_2)$ in two, three and four dimensions are performed analytically. In two dimensions the two-boson scattering amplitude for $M_2 < 2M$ is given by

$$\tau^{-1}(M_2) = \frac{i}{\pi} \left\{ \frac{\operatorname{arccot} [\beta_{2B}]}{M_{2B}^2 \beta_{2B}} - \frac{\operatorname{arccot} [\beta_2]}{M_2^2 \beta_2} \right\}, \quad (10)$$

where

$$\beta_j \equiv \sqrt{\left(\frac{2M}{M_j}\right)^2 - 1} \quad (j = 2, 2B). \quad (11)$$

In three dimensions ($n = 3$), we have

$$\tau^{-1}(M_2) = \frac{i}{8\pi} \left\{ \frac{1}{M_{2B}} \ln \left(\frac{2M + M_{2B}}{2M - M_{2B}} \right) - \frac{1}{M_2} \ln \left(\frac{2M + M_2}{2M - M_2} \right) \right\}; \quad (12)$$

and, in four dimensions ($n = 4$),

$$\tau^{-1}(M_2) = \frac{i}{8\pi^2} [\beta_{2B} \operatorname{arccot} (\beta_{2B}) - \beta_2 \operatorname{arccot} (\beta_2)]. \quad (13)$$

For the present purpose, the bound-state calculation, it is enough to know $\tau(M_2)$ for $M_2 < 2M$. This concludes our discussion of the renormalized zero-range two-boson interaction in the light-front. In case of a general two-boson separable interaction, with $g(M_{02}^2) \neq 1$ in eq. (7), one has to modify correspondingly the other equations for the three-boson vertex function, with the inclusion of the two-body form-factor, as shown in section 3. We also present in the next section the discussion of the phase-space constraints in the light-front formalism of the equations.

3. Light-front reduction of the Bethe-Salpeter equation

The integral equation for v is obtained after the integration of equation (1) over k^- . The integration over k^- is performed with the use of Cauchy theorem, assuming that the vertex is analytic in the lower half of the complex k^- plane. The pole, in the k^- plane, comes from the on-mass-shell condition of the momentum of the spectator particle. This allows direct access to the three-particle Fock-state component of the wave-function. In the corresponding truncated Faddeev equation, only intermediate state propagations with three-particles are allowed. Thus, in the vertex function, the spectator boson is on-mass-shell. The momentum variables in the integral equation, q^+ and q_\perp , are the momentum in the null-plane for an on-mass-shell particle (the transversal momentum is needed in three and four space-time dimensions).

We have started with a contact interaction for the bosons in eq. (1), which leads to a divergence in the transverse momentum integration of the two-boson scattering

amplitude in four space-time dimensions. We also suggested the introduction of a separable interaction with form-factor $g(M_{02})$, which by consistency should be present in the kernel of the three-boson vertex equation due to the rearrangement of the particles. As the form-factor depends on the free mass of the intermediate interacting pair, it is natural to introduce the form-factors as given below:

$$v(y, \vec{p}_\perp) = \frac{i\tau(M_2)}{(2\pi)^{n-1}} \int \frac{dx d^{(n-2)}k_\perp}{x(1-y-x)} \frac{g(M_{02}^A)g(M_{02}^B)v(x, \vec{k}_\perp)}{(M_{3B}^2 - M_{03}^2)}, \quad (14)$$

where

$$\begin{aligned} (M_{02}^A)^2 &= \frac{k_\perp^2 + M^2}{x} + \frac{(P_3 - k - p)_\perp^2 + M^2}{1 - x - y} - (P_3 - p)_\perp^2 \\ (M_{02}^B)^2 &= \frac{p_\perp^2 + M^2}{y} + \frac{(P_3 - k - p)_\perp^2 + M^2}{1 - x - y} - (P_3 - k)_\perp^2. \end{aligned} \quad (15)$$

From now on, our discussions will be limited to the case of contact interaction, with $g(M_{02}^2) = 1$.

Let us discuss the corresponding limits for the variables $y \equiv q^+/M_{3B}$ and q_\perp . Following ref. [3], the mass of the two-boson subsystem is constrained to be real. In two space-time dimensions, only the momentum fraction is enough to describe the spectator boson, such that

$$M_2^2 = (M_{3B} - q^+) \left(M_{3B} - \frac{M^2}{q^+} \right) > 0. \quad (16)$$

The above relation implies in the inequality

$$1 > y > \frac{M^2}{M_{3B}^2}. \quad (17)$$

In three and four space-time dimensions, the range of values of the perpendicular momentum allowed by the reality of the mass of the two-boson subsystem, comes from:

$$M_2^2 = (M_{3B} - q^+) \left(M_{3B} - \frac{q_\perp^2 + M^2}{q^+} \right) - q_\perp^2 > 0. \quad (18)$$

By solving this inequality for q_\perp^2 , we obtain

$$q_\perp^2 < (1 - y)(M_{3B}^2 y - M^2). \quad (19)$$

The limits for y are the same the dimensions $n=2$ and 3:

$$1 > y > \frac{M^2}{M_{3B}^2}, \quad (20)$$

where the lower bound comes from $q_\perp^2 > 0$.

3.1. Light-front Faddeev equation in 1 + 1 space-time dimensions

The equation for the Faddeev component of the vertex in 1 + 1 space-time dimensions is obtained using eq. (10) and the limits given by eq. (17) together with the result of the k^- integration of eq. (1), as explicated in eq. (14) with $g = 1$:

$$v(y) = - \left\{ \frac{2 \operatorname{arccot}(\beta_{2B})}{M_{2B}^2 \beta_{2B}} - \frac{2 \operatorname{arccot}(\beta_2)}{M_2^2 \beta_2} \right\}^{-1} \times \int_{\frac{M^2}{M_{3B}^2}}^{1-y} \frac{dx}{x(1-y-x)} \frac{v(x)}{(M_{3B}^2 - M_{03}^2)}, \quad (21)$$

where M_2 is given by eq. (16), β_2 is defined in eq.(11) and the mass of the virtual three-boson state is:

$$M_{03}^2 = M^2 \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{1-y-x} \right). \quad (22)$$

The dependence of v on q^- is not specified considering that the spectator boson is on mass-shell. q^+ describe the spectator boson propagation.

Let us now study the non-relativistic limit of eq. (21) ($M \rightarrow \infty$). Considering that $M_2 = 2M - E_2$ and $M_{2B} = 2M - E_{2B}$, with E_{2B} the two-boson binding energy, in the limit $M \rightarrow \infty$ the two-boson amplitude is given by:

$$\lim_{M \rightarrow \infty} \tau^{-1} = \frac{i}{8M^{3/2}} \left[\frac{1}{\sqrt{E_2}} - \frac{1}{\sqrt{E_{2B}}} \right]. \quad (23)$$

The non-relativistic limit of the integrand of eq. (21) involves the change in the variable x to k_z :

$$\lim_{M \rightarrow \infty} \frac{dx}{dk_z} = \frac{x}{M}, \quad (24)$$

and

$$\lim_{M \rightarrow \infty} x = \lim_{M \rightarrow \infty} y = \frac{1}{3}. \quad (25)$$

As in eq. (21)

$$M_{03} = \sqrt{q^2 + M^2} + \sqrt{k^2 + M^2} + \sqrt{(q+k)^2 + M^2}, \quad (26)$$

the non-relativistic limit is given by

$$v_{NR}(q) = \frac{2}{\pi \sqrt{M}} \left\{ \frac{1}{\sqrt{E_2}} - \frac{1}{\sqrt{E_{2B}}} \right\}^{-1} \int dk \frac{v_{NR}(k)}{-E_{3B} - \frac{q^2}{2M} - \frac{k^2}{2M} - \frac{(q+k)^2}{2M}}, \quad (27)$$

where $E_{3B} = 3M - M_{3B}$ is the three-boson binding energy. This equation was obtained in ref. [20]. The nonrelativistic binding energy is obtained analytically:

$$E_{3B} = 4E_{2B}. \quad (28)$$

3.2. Light-front Faddeev equation in 2 + 1 space-time dimensions

The corresponding equation for the Faddeev component of the vertex in 2+1 space-time dimensions is formulated from eq. (12) and the limits given by eqs. (19) and (20) together with the result of the k^- integration of eq. (1), as explicated by eq. (14), which results:

$$v(y, \vec{q}_\perp) = \frac{2}{\pi} \left\{ \frac{1}{M_{2B}} \ln \left(\frac{2M + M_{2B}}{2M - M_{2B}} \right) - \frac{1}{M_2} \ln \left(\frac{2M + M_2}{2M - M_2} \right) \right\}^{-1} \\ \times \int_{\frac{M^2}{M_{3B}^2}}^{1-y} \frac{dx}{x(1-y-x)} \int_{-k_\perp^{max}}^{k_\perp^{max}} dk_\perp \frac{v(x, \vec{k}_\perp)}{M_{3B}^2 - M_{03}^2}, \quad (29)$$

where M_2 is given by eq. (18),

$$k_\perp^{max} = \sqrt{(1-x)(M_{3B}^2 x - M^2)}$$

and the mass of the virtual three-boson state is:

$$M_{03}^2 = \frac{k_\perp^2 + M^2}{x} + \frac{q_\perp^2 + M^2}{y} + \frac{(q+k)_\perp^2 + M^2}{1-y-x}. \quad (30)$$

In this case, the non-relativistic limit ($M \rightarrow \infty$) of the two-boson amplitude, eq. (12), is given by

$$\lim_{M \rightarrow \infty} \tau^{-1} = \frac{i}{16\pi M} [\ln(E_2) - \ln(E_{2B})], \quad (31)$$

where $M_{2B} = 2M - E_{2B}$ and $M_2 = 2M - E_2$.

The non-relativistic limit of the integrand of eq. (29) involves the change in the variable x to k_z in the same steps as we did in 1+1 dimensions. The mass of the virtual three-boson system is substituted in eq. (29) by:

$$M_{03} = \sqrt{\vec{q}^2 + M^2} + \sqrt{\vec{k}^2 + M^2} + \sqrt{(\vec{q} + \vec{k})^2 + M^2}. \quad (32)$$

So, the non-relativistic limit is given by

$$v_{NR}(q) = \frac{2}{\pi M} \frac{1}{[\ln(E_{2B}) - \ln(E_2)]} \int d^2 k \frac{v_{NR}(k)}{-E_{3B} - \frac{\vec{q}^2}{2M} - \frac{\vec{k}^2}{2M} - \frac{(\vec{q} + \vec{k})^2}{2M}}, \quad (33)$$

This is the non-relativistic equation derived in ref. [21], with the nonrelativistic binding energy given by:

$$E_{3B} = 16.3 E_{2B}. \quad (34)$$

3.3. Light-front Faddeev equation in 3+1 space-time dimensions

The integral equation for the Faddeev component of the three boson vertex, $v(y, \vec{p}_\perp)$, in the center-of-mass system, is given by :

$$v(y, \vec{p}_\perp) = \frac{1}{\pi} \frac{1}{[\beta_{2B} \operatorname{arccot}(\beta_{2B}) - \beta_2 \operatorname{arccot}(\beta_2)]}$$

$$\times \int_{\frac{M^2}{M_{3B}^2}}^{1-y} \frac{dx}{x(1-y-x)} \int^{k_{\perp}^{max}} d^2 k_{\perp} \frac{v(x, \vec{k}_{\perp})}{M_{3B}^2 - M_{03}^2}. \quad (35)$$

We assume that M_2 , eq. (19), has only real values and, as a consequence, there is a maximum of $k_{\perp}^{max} = \sqrt{(1-x)(M_{3B}^2 x - M^2)}$.

Let us now study the non-relativistic limit of eq. (35) ($M \rightarrow \infty$). In this limit, from eq. (13) we have:

$$\lim_{M \rightarrow \infty} \tau^{-1} = \frac{i}{16\pi\sqrt{M}} \left[\sqrt{E_{2B}} - \sqrt{E_2} \right], \quad (36)$$

where $M_{2B} = 2M - E_{2B}$ and $M_2 = 2M - E_2$.

The non-relativistic limit of the integrand of eq. (35) involves the change in the variable x to k_z in the same steps as we did in 2+1 dimensions. From eqs. (32) and (35),

$$v_{NR}(\vec{q}) = \frac{\sqrt{M}}{\pi^2(\sqrt{E_{2B}} - \sqrt{E_2})} \int \frac{d^3 k}{M^2} \frac{v_{NR}(\vec{k})}{\left[-E_{3B} - \frac{\vec{q}^2}{2M} - \frac{\vec{k}^2}{2M} - \frac{(\vec{q}+\vec{k})^2}{2M} \right]}. \quad (37)$$

In eq. (35), the Faddeev component of the ground state vertex is rotationally symmetric in the perpendicular plane, while in eq. (37) it is spherically symmetric.

The Efimov effect [22] appears in eq. (35). In the Efimov limit, that is reached when the two-body binding energy is zero (infinite scattering length), the nonrelativistic Faddeev equation (37) in the s -wave presents an infinite number of three-body bound-states close to zero. The same limit can be identified with the Thomas collapse \dagger , through a rescaling of the equation (37). In order to solve the eq. (37), we need to use a cutoff Λ in the momentum integration to avoid the collapse of the three-boson system. In units such that $\Lambda = 1$ and $M = 1$, this equation can be written as

$$v_{NR}(\vec{z}) = \frac{1}{\pi^2(\sqrt{\varepsilon_{2B}} - \sqrt{\varepsilon_2})} \int d^3 z' \frac{v_{NR}(\vec{z}') \theta(1 - |\vec{z}'|)}{\left[-\varepsilon_{3B} - \frac{z^2}{2} - \frac{z'^2}{2} - \frac{(\vec{z}+\vec{z}')^2}{2} \right]}, \quad (38)$$

where $\varepsilon_N = E_N/\Lambda^2$ ($N = 2, 2B$ or $3B$). The collapse of the three-boson system occurs for $\Lambda \rightarrow \infty$, when E_2 is fixed. The Efimov effect is manifested for $E_{2B} \rightarrow 0$, when Λ is fixed. So, in the nonrelativistic system both effects are related by a scale transformation and correspond to the same limit $\varepsilon \rightarrow 0$ (see also ref.[24]).

The Efimov effect holds in the relativistic approach considered in this paper, but the short-range behavior is strongly modified: the Thomas collapse [23] of the system is avoided, due to the lower bound of the ground state mass in eq. (35), as it will be discussed in the next section.

\dagger In 1935, Thomas [23] has shown that the triton binding energy will collapse in the zero-range limit of the two-body interaction.

4. Lower bounds to the three-boson binding energy

Returning to the relativistic equations in $1 + 1$, $2 + 1$ and $3 + 1$ dimensions, eqs. (21), (29) and (35), respectively, the lower bound for the mass of the three-boson system is obtained from the limits on the x integration and the condition $y > M^2/M_{3B}^2$, which implies:

$$\frac{M^2}{M_{3B}^2} < 1 - \frac{M^2}{M_{3B}^2}, \quad (39)$$

giving the bound

$$M_{3B} > \sqrt{2}M. \quad (40)$$

The same inequality has been already discussed in $3 + 1$ dimensions in reference [3]. The following inequality holds $M + M_{2B} > M_{3B} > \sqrt{2}M$; and, by using $M_{2B} = 2M - E_{2B}$ and $M_{3B} = 3M - E_{3B}$, we obtain $E_{2B} < E_{3B} < (3 - \sqrt{2})M$. Defining the binding energy in respect to the elastic scattering threshold of one boson and the two-boson bound state, $B_{3B} = M + M_{2B} - M_{3B}$, the lower bound for the ratio between B_{3B} and the two-boson binding energy is given by

$$0 < \frac{B_{3B}}{E_{2B}} < (3 - \sqrt{2})\frac{M}{E_{2B}} - 1. \quad (41)$$

The ratio between the three-boson binding energy and the two-boson binding energy is taken to stress the universal scaling that occurs in the non-relativistic limit, in $1+1$ and $2+1$ space-time dimensions. However, in the relativistic domain, the boson mass is also an energy scale which is evidenced by the lower bound in eq. (41). The numerical results confirm the boson mass as a physical scale.

In $3+1$ space-time dimensions, the non-relativistic limit of the three-boson system, has the collapse of the ground state if no physical cutoff to high momentum intermediate state propagation in eq. (37) exists. The two-boson binding energy in the strict non-relativistic limit is not the only physical scale and the collapse represents the emergence of a three-body scale. However, in the relativistic eq. (35), the boson mass scale matters and the collapse is avoided. As E_{2B} goes to zero the Efimov effect is still present in the relativistic eq. (35), as the boson mass becomes irrelevant as a physical momentum scale in the non-relativistic limit [3]. In nonrelativistic quantum mechanics the Efimov and Thomas effects are equivalent phenomena as one can verify by rescaling the two-body scattering length and interaction range [24], however the relativistic propagation of the bosons breaks the equivalence as the boson mass is a momentum scale that has no effect in the nonrelativistic limit, but avoids the collapse in the relativistic situation [3, 25].

5. Numerical results and conclusion

In figure 4, for $M = 1$, the numerical solution of the light-front Faddeev equations for the ratio between the binding energies of three-boson to the two-boson systems is shown as a function of the two-boson binding energy for $E_{2B} < (3 - \sqrt{2})$ and compared to the lower bound

$(3 - \sqrt{2})/E_{2B} - 1$. In (1+1) dimensions, the non-relativistic limit of eq. (21) for $E_{2B} \ll 1$, results in the nonrelativistic equation for the zero-range potential, eq. (27), which has a solution for $B_{3B}/E_{2B} = 3$ [20], as is observed in figure 4. In (2+1) dimensions, the non-relativistic limit of eq. (29) is the nonrelativistic equation for the zero-range potential, eq. (33), which has a solution for $B_{3B}/E_{2B} = 15.3$ [21], that we can see in figure 4 as E_{2B} approaches zero. In (3+1) dimensions, the light-front three-boson eq. (35), even being formally similar to the nonrelativistic eq. (37), still carries the boson mass scale, due to the momentum constraints in eq. (35), and consequently B_{3B} goes to a finite value as E_{2B} is close to zero. It does not present the Thomas effect. The lower bound of B_{3B} is approached by our calculation in (1+1) and (2+1) dimensions for E_{2B} above 0.4, while it overestimate the binding in (3+1) dimensions by a factor of about 3.

In conclusion, we show how to reduce the ladder Bethe-Salpeter equation for a three-boson system to the null-plane. The light-front dynamics of the three-boson bound state was performed in $1 + 1$, $2 + 1$ and $3 + 1$ space-time dimensions. The two-boson bound-state mass is used to parametrize the mass of the virtual two-body subsystem, in the renormalized two-body amplitude of the relativistic three-body Faddeev equation. With the mass of the virtual two-body subsystem constrained to real values, the momentum phase-space of the spectator particle is delimited, which implies on a lower bound to the three-body ground state energy. We compared the ground state energy obtained from the numerical solution of the light-front Faddeev equations and found that the lower bound is approached by the system ground state energy in (1+1) and (2+1) space-time dimensions in the relativistic region of the two-boson bound state energy, where it attains values around the boson mass. Since the lower bound derivation involves only phase space considerations, it seems possible to extend the present work to three-fermions systems in the light-front interacting with contact interactions.

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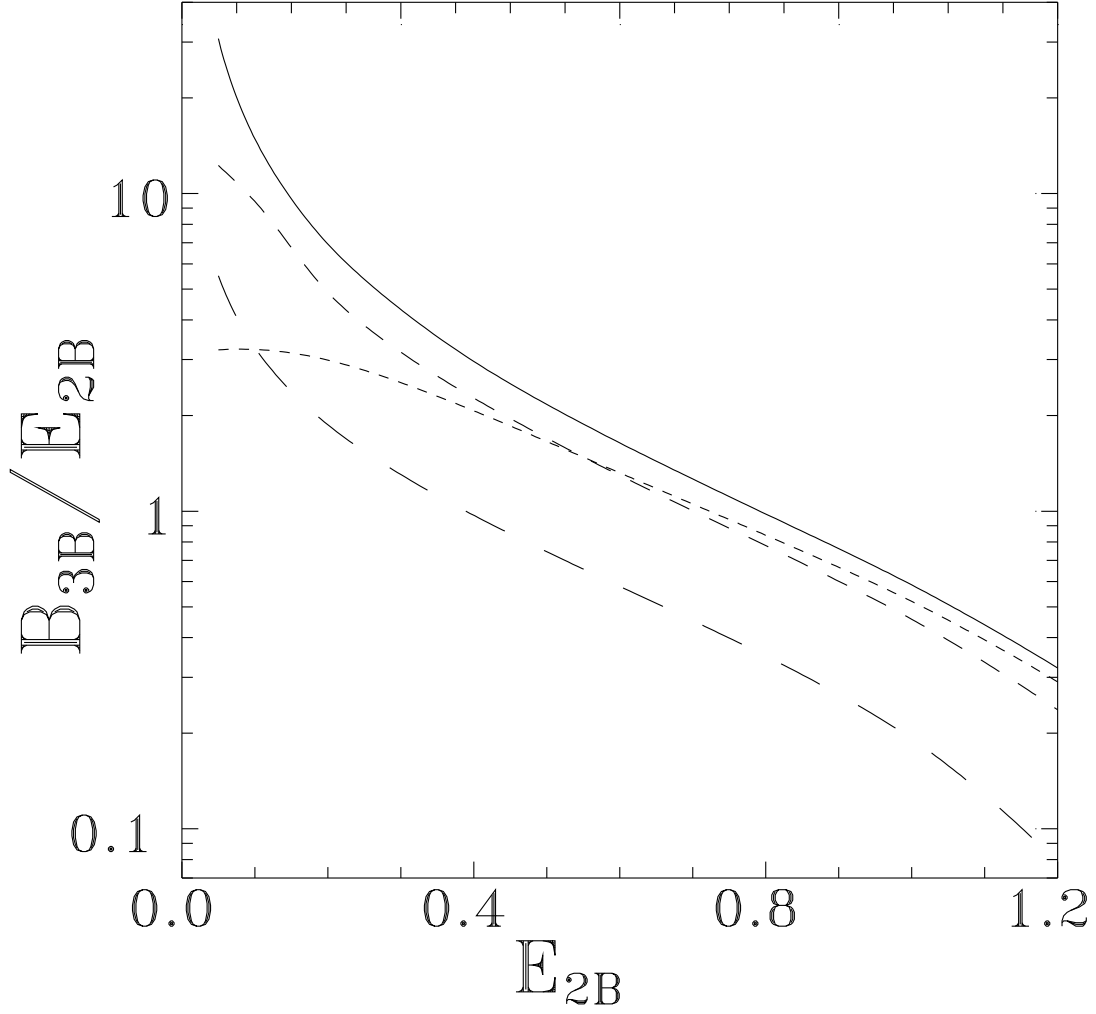


Figure 4. The three-boson binding energy, $B_{3B} = M + M_{2B} - M_{3B}$, is represented as a function of the two-boson mass in units of the single-boson mass, for one (short-dashed-line), two (dashed-line) and three (long-dashed-line) space-time dimensions. The solid curve is the lower bound.

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